Solving Emerson-Lei Games via Zielonka Trees

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Overview

- Winning regions in various ω-regular games are nested fixpoints.
- Emerson-Lei objectives succinctly encode standard objectives.
- Zielonka trees characterize winning in Emerson-Lei games.

We show how to extract a nested fixpoint from any Zielonka tree, resulting in a *symbolic* algorithm that solves Emerson-Lei games with n nodes, m edges and k colors in time $\mathcal{O}(k! \cdot m \cdot n^{\frac{k}{2}})$.

This generalizes previous fixpoint algorithms for Büchi, parity, GR[1], Rabin and Streett games, recovering previous upper bounds on runtime.

Emerson-Lei Games

Infinite-duration zero-sum games played by two players $\exists$ and $\forall$:

$$G = (V = V_\exists \cup V_\forall, E \subseteq V \times V, \mathsf{col} : V \to 2^C, \varphi) \qquad \varphi \in \mathbb{B}(\mathsf{GF}(C))$$

Player $\exists$ wins play $\pi \subseteq V^\omega$ in G if and only if $\mathsf{col}[\pi] \models \varphi$

Examples:

$$\varphi = \mathsf{GF}\, f \qquad \text{(Büchi)}$$

$$\varphi = \bigwedge_{1\leq i\leq k} \mathsf{GF}\, f_i \qquad \text{(gen. Büchi)}$$

$$\varphi = \bigwedge_{1\leq i\leq k} \mathsf{GF}\, p_i \to \bigwedge_{1\leq j\leq k} \mathsf{GF}\, q_j \qquad \text{(GR[1])}$$

$$\varphi = \bigvee_{i \text{ even}} \mathsf{GF}\, p_i \wedge \mathsf{FG} \bigwedge_{i<j\leq k} \neg p_j \qquad \text{(parity)}$$

$$\varphi = \bigvee_{1\leq i\leq k} \mathsf{GF}\, e_i \wedge \mathsf{FG}\, \neg f_i \qquad \text{(Rabin)}$$

$$\varphi = \bigwedge_{1\leq i\leq k} (\mathsf{GF}\, r_i \to \mathsf{GF}\, g_i) \qquad \text{(Streett)}$$

$$\varphi = \bigvee_{U\in\mathcal{U}} \bigwedge_{i\in U} \mathsf{GF}\, f_i \wedge \mathsf{FG} \bigwedge_{j\notin U} f_j \qquad \text{(Muller for } \mathcal{U} \subseteq 2^C)$$

Emerson-Lei games are determined, but not positional (e.g. Streett).

Zielonka Trees

Tree $\mathcal{Z}_\varphi$ with vertices X labeled by $l(X) \subseteq C$, subject to certain maximality conditions. Vertex X is *green* if $l(X) \models \varphi$ and *red* otherwise.

Require for all children Y, Y' of X:
X green $\Leftrightarrow$ Y red, $l(Y) \subsetneq l(X)$, $l(Y)$ and $l(Y')$ are incomparable.

Lemma: The Zielonka tree $\mathcal{Z}_\varphi$ has at most $e \cdot |C|!$ vertices.

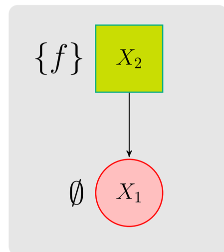
Play $\pi = v_0 v_1 \dots$ induces *walk* ρ_π through Zielonka tree:

- start with v_0 and left-most leaf in Zielonka tree;
- at v_i and X, pick lowest ancestor Y of X s.t. $\mathsf{col}(v_i) \subseteq l(Y)$ and proceed with v_{i+1} and left-most leaf X' under Y that is to right of X

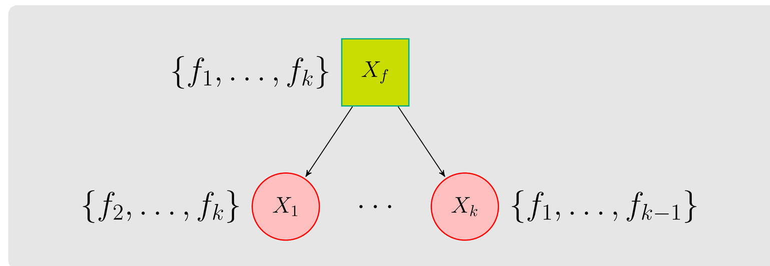
Dominating vertex: topmost node that is seen infinitely often in ρ_π.

Lemma: Player $\exists$ wins play $\pi \Leftrightarrow$ dominating vertex in ρ_π is green.

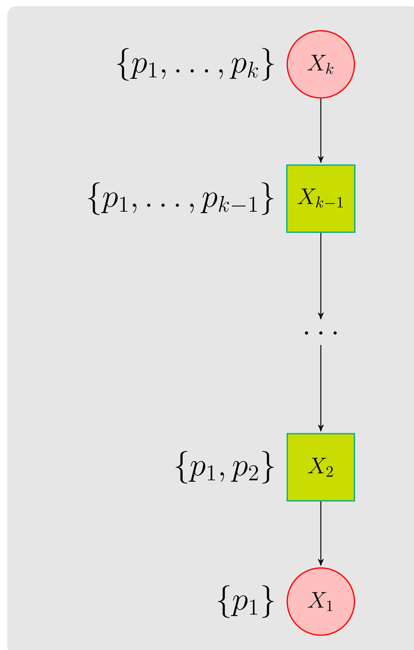
Zielonka Trees by Example



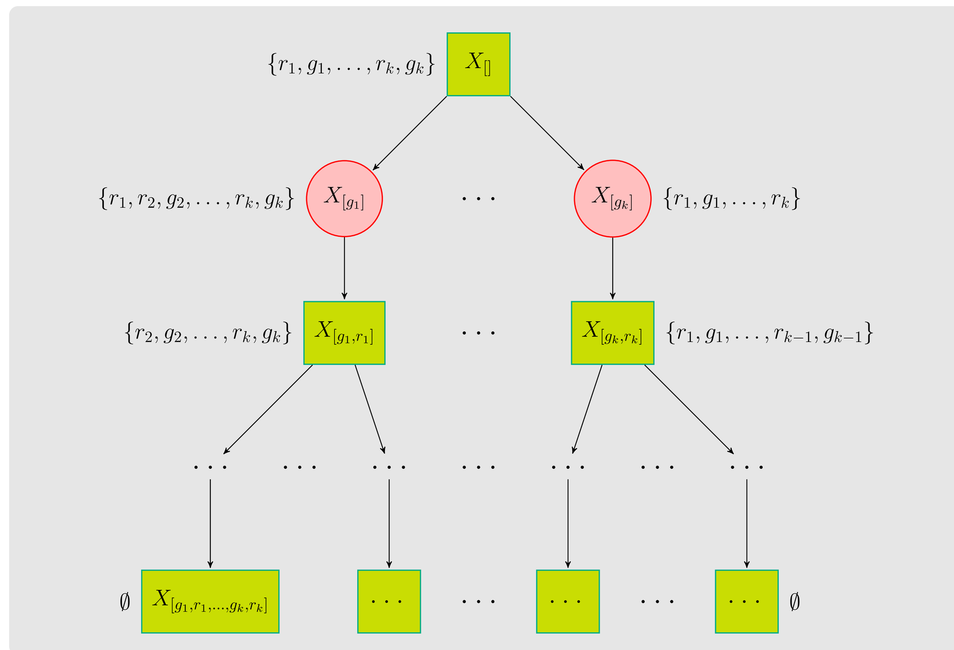
Büchi objective



generalized Büchi objective



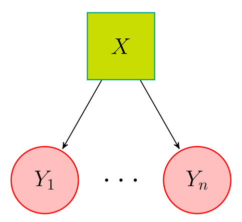
parity objective



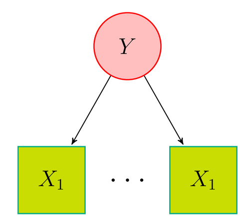
Streett objective

Fixpoint Extraction – Building Blocks

Inner vertices:

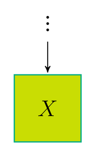


$\rightsquigarrow \; X =_\nu Y_1 \cap \ldots \cap Y_n$

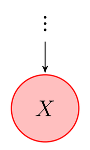


$\rightsquigarrow \; Y =_\mu X_1 \cup \ldots \cup X_n$

Leaf vertices:



$\rightsquigarrow \; X =_\nu \bigcup_{Y\geq X} \mathsf{Walk}(X,Y) \cap \mathsf{Cpre}(Y)$



$\rightsquigarrow \; X =_\mu \bigcup_{Y\geq X} \mathsf{Walk}(X,Y) \cap \mathsf{Cpre}(Y)$

where

$$\mathsf{Walk}(X,Y) = \{v \in V \mid Y \text{ is lowest ancestor of } X \text{ s.t. } \mathsf{col}(v) \subseteq l(Y)\}$$

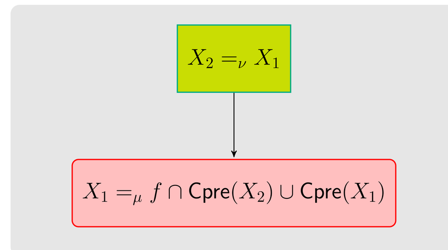
for vertices X, Y, and Cpre encodes one-step attraction for player $\exists$.

Main Result

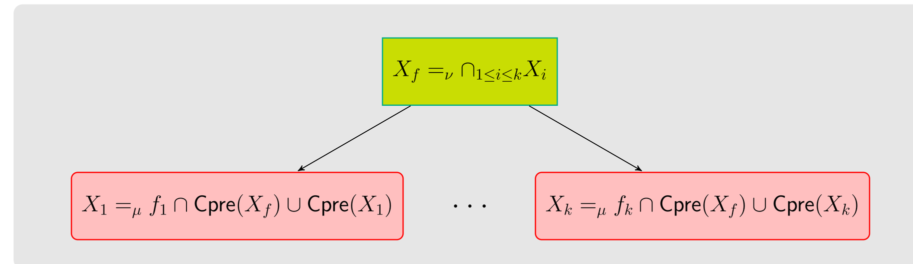
Theorem: The solution of the extracted fixpoint equation system is the winning region in the corresponding Emerson-Lei game.

$\Rightarrow$ Solve equation systems by fixpoint iteration to solve Emerson-Lei games with n nodes and k colors symbolically in time $\mathcal{O}(k! \cdot n^{\frac{k}{2}+2})$. For simpler conditions, this recovers previous fixpoint iteration algorithms.

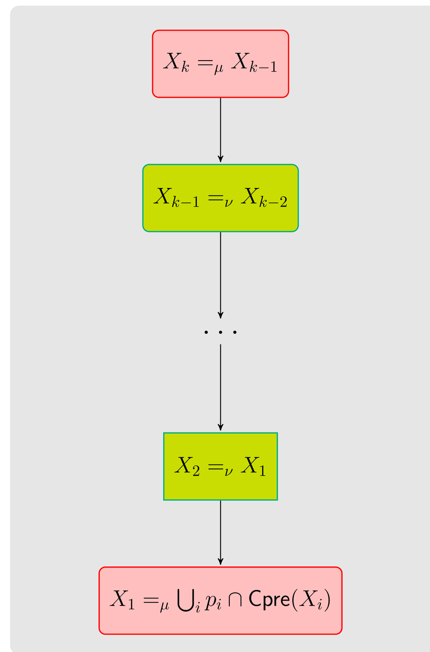
Extracted Fixpoint Systems by Example



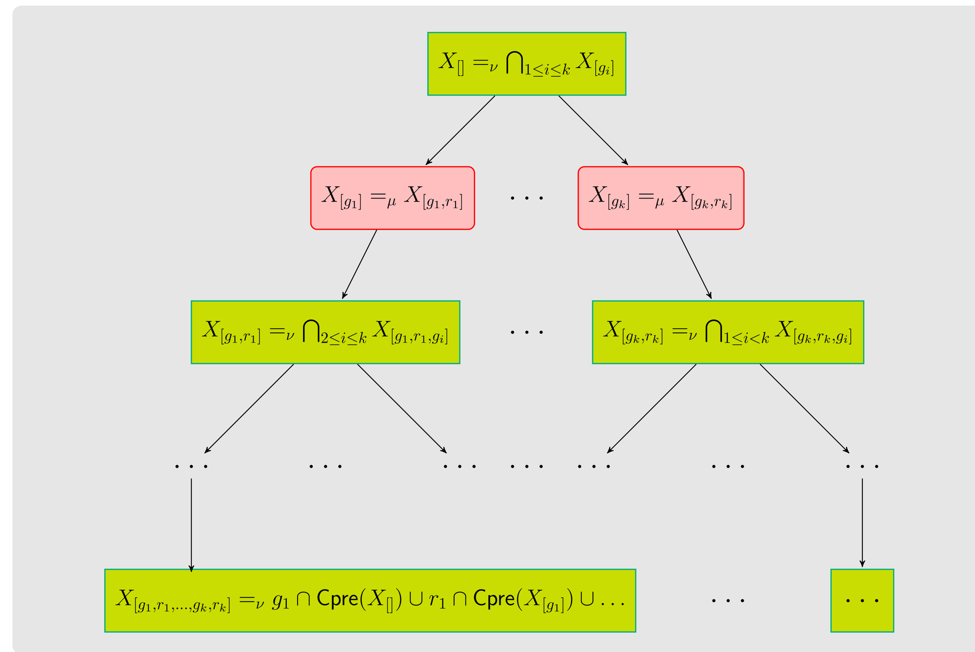
Büchi objective



generalized Büchi objective



parity objective



Streett objective

Future Work

- Use universal trees to solve equation systems in time $\mathcal{O}(k! \cdot n^{\log k})$, generalizing quasipolynomial method to Emerson-Lei games.
- Implement solving algorithm, finds direct application in reactive synthesis for the *Emerson-Lei and safety fragment* of LTL.
- Similar reduction from alternating Emerson-Lei automata to alternating weak automata.

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More details and results: https://arxiv.org/pdf/2305.02793.pdf