Solving Emerson-Lei Games via Zielonka Trees

A Symbolic Fixpoint Algorithm

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Highlights, July 11, 2023

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European Research Council
Established by the European Commission

Main Result

- Symbolic algorithm for solving Emerson-Lei games directly
- Emerson-Lei objective $\to$ Zielonka Tree $\to$ Fixpoint Equation System
- Runtime $\mathcal{O}(k! \cdot n^{\frac{k}{2}}) \in 2^{\mathcal{O}(k \log n)}$ versus $2^{\mathcal{O}(kn)}$ (LAR[1])

[1] [Dawar, Hunter, 2005]

Emerson-Lei Objectives

Emerson-Lei Games

$$G = (V, E \subseteq V \times V, \mathsf{col} : V \to 2^{\mathsf{C}}, \varphi) \qquad \varphi \in \mathbb{B}(\mathsf{GF}(\mathsf{C}))$$

Player $\exists$ wins play π iff $\mathsf{col}[\pi] \models \varphi$

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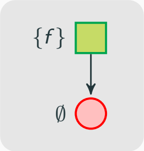
Examples:

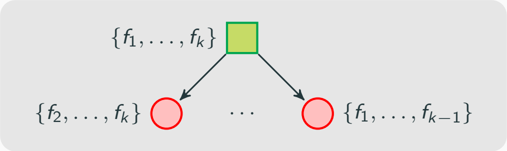
$C = \{f\}$	$\varphi = \mathsf{GF}\, f$	(Büchi)
$C = \{f_1, \ldots, f_k\}$	$\varphi = \bigwedge_{1 \leq i \leq k} \mathsf{GF}\, f_i$	(gen. Büchi)
$C = \{p_1, \ldots, p_{2k}\}$	$\varphi = \bigvee_{i \text{ even}} \mathsf{GF}\, p_i \wedge \bigwedge_{j > i} \mathsf{FG}\, \neg p_j$	(parity)
$C = \{e_1, f_1, \ldots, e_k, f_k\}$	$\varphi = \bigvee_{1 \leq i \leq k} \mathsf{GF}\, e_i \wedge \mathsf{FG}\, \neg f_i$	(Rabin)
$C = \{r_1, g_1, \ldots, r_k, g_k\}$	$\varphi = \bigwedge_{1 \leq i \leq k} \mathsf{GF}\, r_i \to \mathsf{GF}\, g_i$	(Streett)

Determined, not positional (in general: memory $|C|!$)

Zielonka Trees by Example



Büchi objective



generalized Büchi objective

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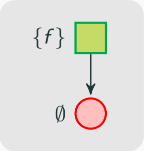

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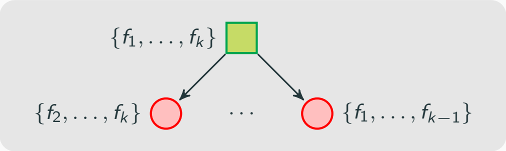

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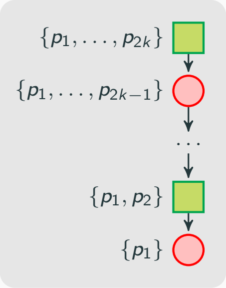
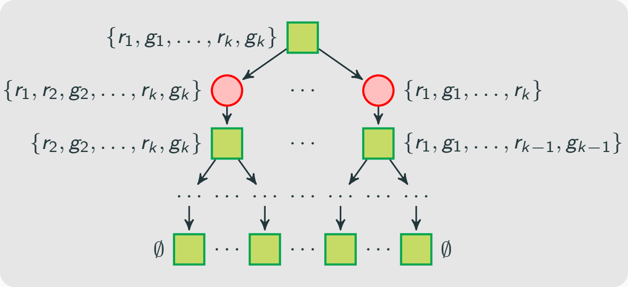


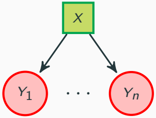
parity objective



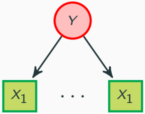
Streett objective

- Extract system of fixpoint equations over V
- One equation per vertex (at most $e \cdot |C|!$), alternation-depth $|C|$

Building blocks:



$\rightsquigarrow X =_\nu Y_1 \cap \ldots \cap Y_n$



$\rightsquigarrow Y =_\mu X_1 \cup \ldots \cup X_n$

$$\rightsquigarrow Z =_\nu \bigcup_{Z_i \geq Z} \mathrm{col}^{-1}_{Z,Z_i} \cap \mathrm{Cpre}(Z_i)$$

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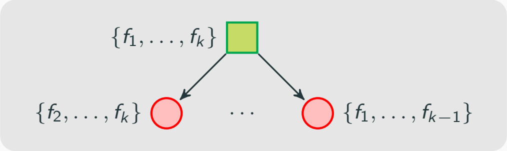
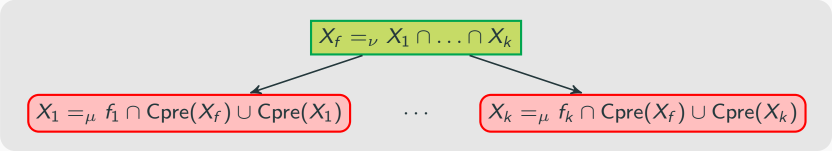
Fixpoint Extraction by Example

Generalized Büchi objective:



$$\Downarrow$$

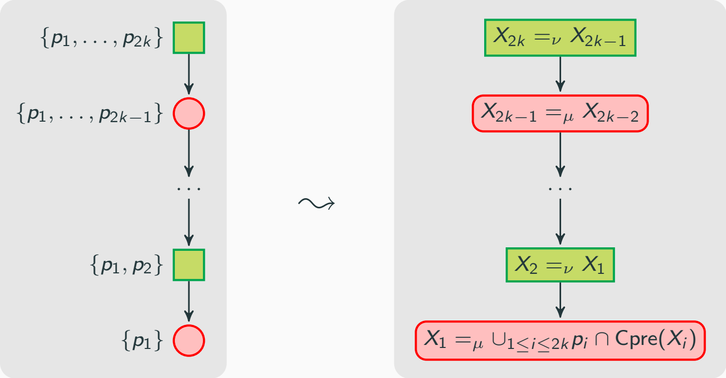


$$=$$

$$\nu X_f.\,(\mu X_1.\, f_1 \cap \mathsf{Cpre}(X_f) \cup \mathsf{Cpre}(X_1)) \cap \ldots \cap$$
$$(\mu X_k.\, f_k \cap \mathsf{Cpre}(X_f) \cup \mathsf{Cpre}(X_k))$$

Fixpoint Extraction by Example

Parity objective:



$$=$$

$$\nu X_{2k}.\, \mu X_{2k-1}.\, \ldots .\, \nu X_2.\, \mu X_1.\, \bigcup_{1 \leq i \leq 2k} p_i \cap \mathsf{Cpre}(X_i)$$

Ongoing Work: Universal Trees for Emerson-Lei Games

Solving equation system by fixpoint iteration: $\mathcal{O}(k!n^{\frac{k}{2}})$

Ongoing work:

- Tight bounds on universal tree size beyond parity objectives; conjecture: $\mathcal{O}(k!n^{\log k})$

Summary

Take-away:

- Direct fixpoint characterization of Zielonka trees
- Adaptive fixpoint algorithm for Emerson-Lei games
- Solves Emerson-Lei games with n nodes, k colors in time $\mathcal{O}(k!n^{\frac{k}{2}})$

Application: Symbolic reactive synthesis for EL+safety fragment of LTL