Faster Game Solving by Fixpoint Acceleration

Daniel Hausmann
Gothenburg University, Sweden

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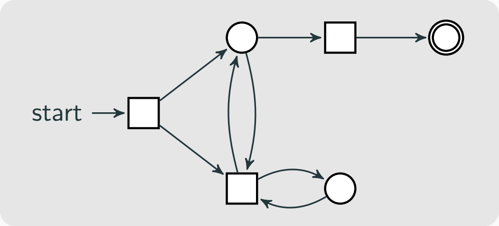
Two-Player Games

Games: algorithmic essence of verification, reasoning, synthesis, . . .



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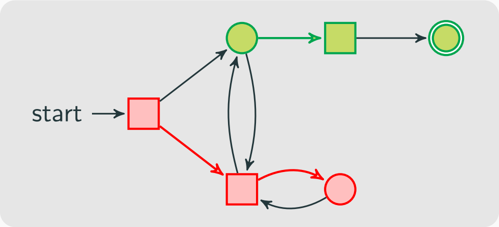

- How to compute *winning regions* ($\text{win}_\exists$, $\text{win}_\forall$)?
- How to extract *winning strategies*?
- *Reduction* of problems to game solving

Parity Games

Parity Games

$G = (V, E \subseteq V \times V, \Omega : V \to \{1, \dots, 2k\})$

- play: $\pi = v_0 v_1 \ldots \in V^\omega$ with $(v_i, v_{i+1}) \in E$ for all $i \geq 0$
- player $\exists$ wins play π iff $\max(\mathrm{Inf}(\Omega[\pi]))$ is even

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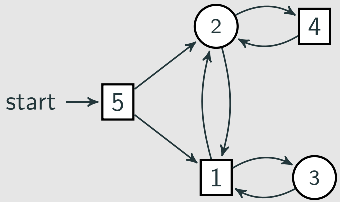

- Parity games are positionally determined
- Solving parity games is in QP and in NP $\cap$ co-NP

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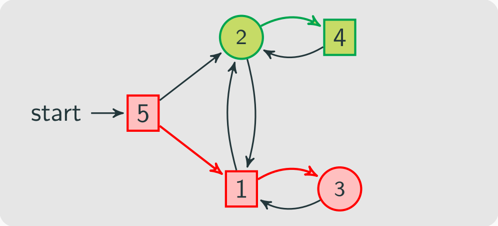

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Extremal fixpoints of monotone $f : \mathcal{P}(U) \to \mathcal{P}(U)$ for finite set U:

$$
\mu X.\, f(X) = \bigcap \{ W \subseteq U \mid f(W) \subseteq W \} = f^{|U|}(\emptyset)
$$
$$
\nu X.\, f(X) = \bigcup \{ W \subseteq U \mid W \subseteq f(W) \} = f^{|U|}(U)
$$

Fixpoint Characterization of Winning, reachability

Reachability game $G = (V, E \subseteq V^2, F)$, $V = V_\exists \mathbin{\dot\cup} V_\forall$

Controllable predecessor function (one-step forcing):

$$\begin{aligned}\mathrm{CPre}(X) = &\{v \in V_\exists \mid \exists (v,w) \in E.\, w \in X\} \cup \\ &\{v \in V_\forall \mid \forall (v,w) \in E.\, w \in X\}\end{aligned}$$

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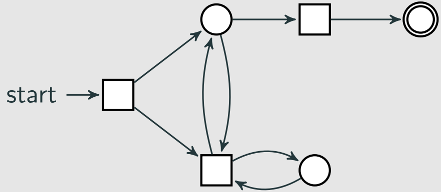

$$\begin{aligned}\mathrm{win}_\exists &= F \cup \mathrm{CPre}(F) \cup \mathrm{CPre}(\mathrm{CPre}(F)) \cup \ldots \\ &= \mu X.\, (F \cup \mathrm{CPre}(X))\end{aligned}$$

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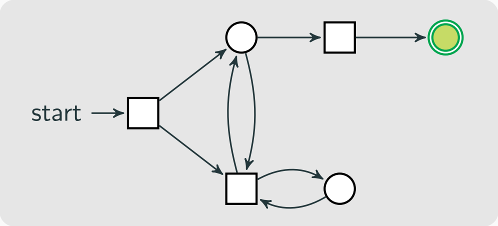

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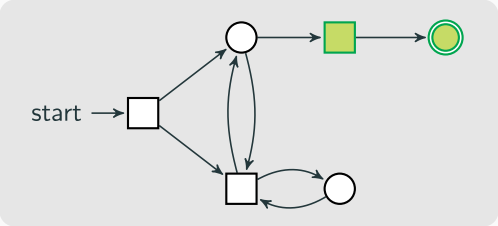

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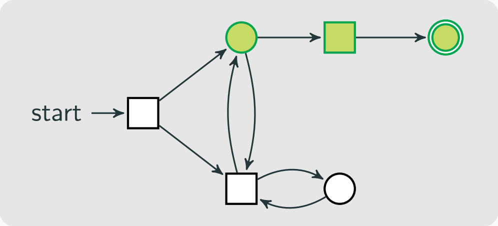

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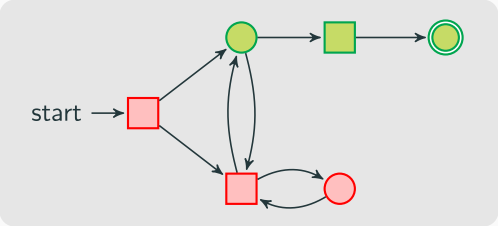

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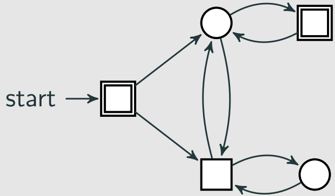

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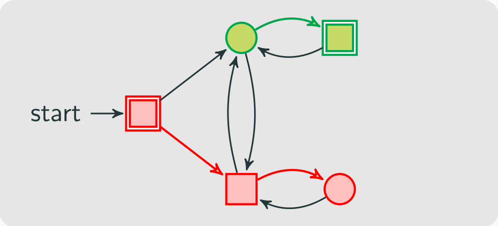

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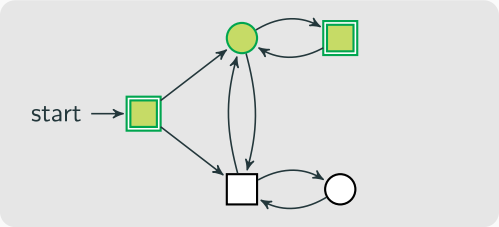

$$\text{win}_\exists = \nu X.\, \mu Y.\, (F \cap \mathsf{CPre}(X)) \cup \mathsf{CPre}(Y) = Y^{|V|}(V)$$

$$Y^1(V) = \mu Y.\, (F \cap \mathsf{CPre}(V)) \cup \mathsf{CPre}(Y)$$

...

...

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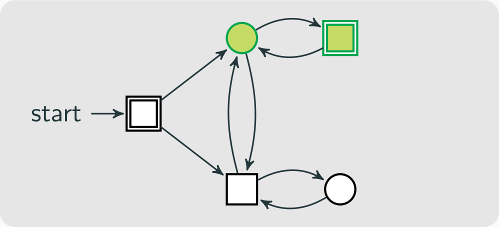

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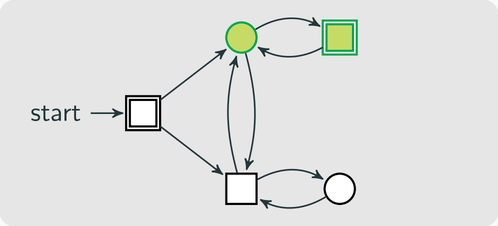

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$$Y^3(V) = Y^2(V)$$

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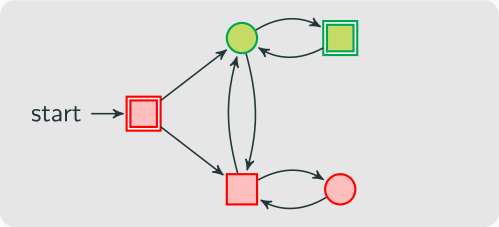

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Fixpoint Characterization of Winning, parity

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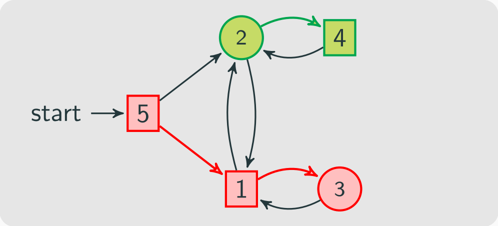
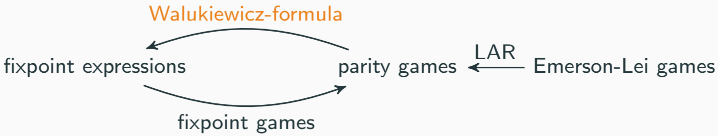


Walukiewicz-formula (writing $\Omega_i = \{v \in V \mid \Omega(v) = i\}$):

$$\mathrm{win}_\exists = \nu X_{2k}.\, \mu X_{2k-1}. \ldots . \nu X_2.\, \mu X_1. \bigcup_{1 \leq i \leq 2k} \Omega_i \cap \mathrm{CPre}(X_i)$$

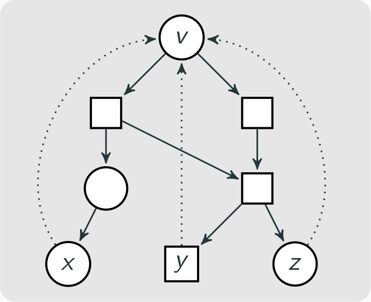
Accelerated Game Solution

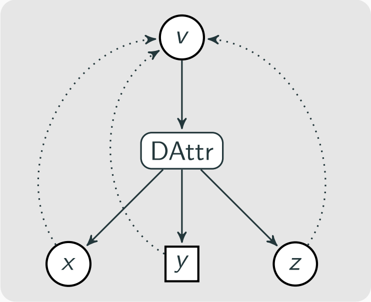


- Adapt Walukiewicz-formulas to use multi-step attraction (DAttr) in place of one-step attraction (Cpre)
- Shrinks domain of fixpoint computations $\rightsquigarrow$ faster game solving

Accelerated Solution of DAG Parts in Games

n nodes, m non-DAG nodes





$\nu X.\,\mathsf{CPre}(X)$

n iterations of CPre

$\nu Y.\,\mathsf{DAttr}(Y)$

m iterations of DAttr

Main Result

Fix parity game $G = (V, E, \Omega : V \to \{1, \ldots, d\})$ with DAG nodes W

DAG attractor (to $Z \subseteq V \setminus W$)

Region from where player $\exists$ can force exiting W to Z:

$$\mathsf{DAttr}_W(Z) = \mu X.\, Z \cup (W \cap \mathsf{CPre}(X))$$

$m := |V| - |W|$

Theorem

G can be solved with $\mathcal{O}(m^{\log d})$ computations of a DAG attractor.

Advantageous if $m < \log n$ and DAG attraction can be checked efficiently

Fixpoint of DAG attraction

Replace

$$\nu X_{2k}.\,\mu X_{2k-1}.\,\ldots.\,\nu X_2.\,\mu X_1.\bigcup_{1\leq i\leq 2k}\Omega_i\cap\mathsf{CPre}(X_i)$$

with

$$\nu Y_{2k}.\,\mu Y_{2k-1}.\,\ldots.\,\nu Y_2.\,\mu Y_1.\,\mathsf{DAttr}_W(Y_1,\ldots,Y_k)$$

The former lives over V, the latter over $V\setminus W$

Examples

Particularly helpful for games that encode predicate $f : 2^V \to 2^V$:

assume $V_\forall = \mathcal{P}(V_\exists)$ and

- $\exists$ can move from v to $U \subseteq V$ s.t. $v \in f(U)$
- $\forall$ can move from $U \subseteq V$ to $u \in U$

$\rightsquigarrow$ DAGs of size $2^{|V|}$; faster game solving if f can be evaluated efficiently

Examples of games of this shape

- Model checking generic μ-calculi [CONCUR 2019, VMCAI 2024]
- Satisfiability checking generic μ-calculi [FoSSaCS 2019, CADE 2023]
- Baldan, König, Padoan: Solution of fixpoint games [POPL 2018]

Accelerated Game Solution

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- Later-appearance record (LAR) reduction preserves DAG sub-games

Emerson-Lei Games

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$$G = (V, E \subseteq V \times V, \mathsf{col} : V \to 2^{\mathsf{C}}, \varphi) \qquad \varphi \in \mathbb{B}(\mathsf{GF}(\mathsf{C}))$$

Player $\exists$ wins play π iff $\mathsf{col}[\pi] \models \varphi$

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Examples:

$$\begin{array}{lll} C = \{f\} & \varphi = \mathsf{GF}\, f & \text{(Büchi)} \\ C = \{p_1, \ldots, p_{2k}\} & \varphi = \bigvee_{i \text{ even}} \mathsf{GF}\, p_i \wedge \bigwedge_{j>i} \mathsf{FG}\, \neg p_j & \text{(parity)} \\ C = \{e_1, f_1, \ldots, e_k, f_k\} & \varphi = \bigvee_{1 \le i \le k} \mathsf{GF}\, e_i \wedge \mathsf{FG}\, \neg f_i & \text{(Rabin)} \\ C = \{r_1, g_1, \ldots, r_k, g_k\} & \varphi = \bigwedge_{1 \le i \le k} \mathsf{GF}\, r_i \to \mathsf{GF}\, g_i & \text{(Streett)} \end{array}$$

Determined, not positional (in general: memory $|C|!$)

LAR Reduction

Later-appearence-record (LAR) reduction: transforms Emerson-Lei game with d colors to parity game with $2d$ priorities; blow-up on state space: $d!$

Theorem

LAR reduction preserves DAG structure.

Fix Emerson-Lei game with d colors, DAG nodes W, $m := |V| \setminus |W|$

Corollary

G can be solved with $\mathcal{O}((m \cdot d!)^{\log d})$ computations of DAG attractor.

Summary

Take-away:

- Winning regions in games are fixpoints of one-step forcing function
- Replace one-step forcing function with multi-step DAG attraction
- Method works for parity games, extends to Emerson-Lei games

- Assumes given partition into DAG and non-DAG parts
- $\mathcal{O}(n^{\log d})$ iterations of one-step attraction vs.
  $\mathcal{O}(m^{\log d})$ iterations of DAG attraction
- Helps if $m < \log n$ and DAG attraction can be computed efficiently